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Discrete Subgroups of $PU(1,2;\mathbf{C})$

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Our aim of this paper is to introduce the following Basmajian-Miner's theorem.

Theorem 1. *Fix a stable basin point (r,r,ε). Let g be a parabolic element with fixed point ∞. If f is a loxodromic element with attracting fixed point 0 and repelling fixed point q satisfying*

$$|\lambda(f)-1|<\varepsilon$$

and

$$\delta(0,q)\geq\frac{\delta(0,g(0))}{r^2}(1+r^2+\sqrt{1+r^2}),$$

then the group $<f,g>$ generated by f and g is not discrete.

1. Let $H^2_{\mathbf{C}}$ be complex hyperbolic 2-space. Set p_∞ to be the point $(0,-1,1)$ in the boundary $\partial H^2_{\mathbf{C}}$ of $H^2_{\mathbf{C}}$. Since the Heisenberg group acts simply transitively on $\partial H^2_{\mathbf{C}}-\{p_\infty\}$, we may identify the boundary $\partial H^2_{\mathbf{C}}$ with the one-point compactification of the Heisenberg group. We define the map $\phi:\mathbf{C}\times\mathbf{R}\rightarrow\partial H^2_{\mathbf{C}}-\{p_\infty\}$ by

$$\phi(w,t)=(\frac{2w}{1+|w|^2-it},\frac{1-|w|^2+it}{1+|w|^2-it},1).$$

We extend ϕ to $\mathbf{C}\times\mathbf{R}\cup\{\infty\}$ by setting $\phi(\infty)=(0,-1,1)$. This map ϕ defines Heisenberg coordinates (w,t) on $\partial H^2_{\mathbf{C}}$. The space $\partial H^2_{\mathbf{C}}$ with Heisenberg coordinates is called Heisenberg space and is denoted by H_3. Set $\tilde{H}_3=H_3\cup\{\infty\}$. Under this identification, the action of $PU(1,2;\mathbf{C})$ can be transported to that on $\tilde{H}_3$. We consider the action on $\tilde{H}_3$ of elements of $PU(1,2;\mathbf{C})$ with fixed point ∞. *Translation* by (a,y) is given by

$$T_{(a,y)}(w,t)=(w+a,t+y+2Im(a\overline{w})),$$

where $a\in\mathbf{C}$, $y\in\mathbf{R}$. Note that $T^{-1}_{(a,y)}=T_{(-a,-y)}$. *Rotation* is of the form

$$(w,t)\longmapsto(e^{i\theta}w,t),$$

and (real) dilations look like

$$(w,t)\longmapsto(\lambda w,\lambda^2t),$$

where $\lambda>0$. We say that an element is a *complex dilation* if it is the product of a dilation and a rotation. If g is loxodromic, then it is conjugate to a unique complex dilation with attracting fixed point at ∞ and repelling fixed point at the origin 0, namely $(w,t)\longmapsto(\lambda w,|\lambda|^2t)$. We assume that $|\lambda(g)|>1$.

2. For $p = (w, t_1)$ and $q = (w', t_2)$ in H_3 we define the *Cygan metric* $\delta(p, q)$ by

$$\delta(p,q) = [|w - w'|^4 + \{t_1 - t_2 + 2Im(w\overline{w}')\}^2]^{\frac{1}{4}}.$$

Let B_s denote the ball of radius s on the boundary $\partial H^2_{\mathbf{C}}$ with respect to the Cygan metric. For $0 < r < 1$, the pair of open sets $(B_r, \overline{B}^c_{1/r})$ is said to be *stable* with respect to a set of elements S in $PU(1,2;\mathbf{C})$ if any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}^c_{1/r}.$$

Let $S(r, \varepsilon)$ denote the family of elements conjugate to complex dilation g with fixed points in B_r and $\iota(B_r) = \overline{B}^c_{1/r}$, and satisfying $|\lambda(g) - 1| < \varepsilon$, where $\lambda(g)$ is the complex dilation factor of g and $|\lambda(g)| > 1$. Note that $S(r, \varepsilon)$ is closed under conjugation by the inversion ι.

For positive real numbers r and r' with $r < 1/2$, we define $\varepsilon(r, r')$ by

$$(*) \qquad \varepsilon(r, r') = \sup_{\alpha} \min\{\alpha, \varepsilon(r, r', \alpha)\},$$

where

$$\varepsilon(r, r', \alpha) = \sqrt{2 + \left(\frac{1 - (4+\alpha)r^2)}{1 - (3+\alpha)r^2}\right)^2 \left(\frac{1 - 2r^2}{1 - r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - \sqrt{2},$$

and the supremum is over all real numbers α satisfying

$$\alpha < \frac{1 - 4r^2}{2r^2}.$$

Lemma 2 (Stable Basin Theorem). *Given positive real numbers r and r' with $r < 1/2$, the pair of open sets $(B_{r'}, \overline{B}^c_{1/r'})$ is stable with respect to the family $S(r, \varepsilon(r, r'))$, where $\varepsilon(r, r')$ is given by (*). Furthermore, if $g \in S(r, \varepsilon(r, r'))$, then $\delta(0, g(0)) < \delta(0, a_g)$, where a_g is any fixed point of g.*

Sketch of the Proof. Set $s = 1/r$. Let g be an element conjugate to a complex dilation with an attracting fixed point $a_g \in B_r$ and a repelling fixed point $r_g \in \overline{B}^c_s$.

It is seen that

$$\begin{aligned}
\delta(0, g(0)) &= \delta(0, h^{-1}\tilde{g}h(0)) \\
&= \delta(0, TG\tilde{g}G^{-1}T^{-1}(0)) \\
&= \delta(T^{-1}(0), G\tilde{g}G^{-1}T^{-1}(0)) \\
&\leq k\delta(G^{-1}(T^{-1}(0)), G^{-1}(G\tilde{g}G^{-1}T^{-1}(0))) \\
&\leq k\delta(h(0), \tilde{g}h(0)) \leq k\frac{r'}{k} = r',
\end{aligned}$$

where k depends on r and r'.

We explain the above proof more precisely. Since $S(r,\varepsilon(r,r'))$ is closed under conjugation by ι (inversion), it sufficies to determine conditions which guarantee that $g(0)\in B_{r'}$ for all $g\in S(r,\varepsilon(r,r'))$ in order to establish the pair $(B_{r'},\overline{B}^c_{1/r'})$ is stable under $S(r,\varepsilon(r,r'))$. We may assume that $g(0)\neq 0$. In particular, $a_g\neq 0$.

(1) $h=h(g)$: "normalizing element"; $a_g\mapsto 0,\quad r_g\mapsto\infty$.

(2) $h=G^{-1}_{(\gamma,y)}T^{-1}_{a_g}$, where

T_{a_g}: Heisenberg translation; $0\mapsto a_g$,

$G_{(\gamma,y)}$: parabolic element with fixed point 0, $G_{(\gamma,y)}(\infty)=\iota(\gamma,y)=T^{-1}_{a_g}(r_g)$.

(3) $(\gamma,y)=\iota(T^{-1}_{a_g}(r_g))\in B_{\frac{1}{s-r}}$.

(4) $\tilde{g}=hgh^{-1};\quad \tilde{g}(0)=0,\tilde{g}(\infty)=\infty$.

To simplify notation, set $T=T_{a_g}$ and $G=G_{(\gamma,y)}$.

(5)

$$\begin{aligned}\delta(0,g(0))&=\delta(0,h^{-1}\tilde{g}h(0))\\&=\delta(0,TG\tilde{g}G^{-1}T^{-1}(0))\\&=\delta(T^{-1}(0),G\tilde{g}G^{-1}T^{-1}(0)).\end{aligned}$$

We estimate how much G^{-1} distorts the distance from $T^{-1}(0)$ to $G\tilde{g}G^{-1}T^{-1}(0)$.

(6) $T^{-1}(0)\in B_r$.

(7) $G\tilde{g}G^{-1}(0)=0, G\tilde{g}G^{-1}(\infty)=G(\infty)$.

Assume that there exists a parameter $\alpha>0$, for which $||\lambda(g)|-1|<\alpha$ and $r<\frac{1}{\sqrt{3+\alpha}}$.

(8) $\delta(0,G(\infty))=\delta(0,\iota(\gamma,y))>\frac{1}{s-r}=\frac{1-r^2}{r}$.

(9) $G\tilde{g}G^{-1}(T^{-1}(0))\in B_l$, where $l=\frac{(1+\alpha)(1-r^2)r}{1-(3+\alpha)r^2}$.

(10) Since $l\geq r$, $T^{-1}(0),\quad G\tilde{g}G^{-1}T^{-1}(0)\in B_l$.

(11) $\delta(0,g(0))=\delta(T^{-1}(0),G\tilde{g}G^{-1}T^{-1}(0))\leq k\delta(h(0),\tilde{g}h(0))$.

Next we estimate $\delta(h(0)),\tilde{g}h(0))$.

(12) $h(0)=G^{-1}T^{-1}(0)=G^{-1}(-a_g)$.

(13) $h(0)\in B_{\left(\frac{1-r^2}{1-2r^2}\right)\delta(0,a_g)}$.

(14) $\delta(h(0),\tilde{g}h(0))<\frac{r'}{k}$.

If $r<1/2$ and $\varepsilon<\varepsilon(r,r')$, a triple of non-negative numbers (r,r',ε) is called a *basin point*. If $r'\leq r$, we call (r,r',ε) a *stable basin point*.

Define the *real cross ratio* $|[q_1,q_2,q_3,,q_4)]|$ by

$$|[q_1,q_2,q_3,,q_4]|=\frac{\delta^2(q_3,q_1)\delta^2(q_4,q_2)}{\delta^2(q_4,q_1)\delta^2(q_3,q_2)}.$$

It is easy to show that this real cross ratio is invariant under $PU(1,2;\mathbf{C})$.

Lemma 3. *Suppose f and g are conjugate to complex dilations with fixed points $\{q_1, q_2\}, \{q_3, q_4\}$, respectively. If necessary, interchange the roles of q_3 and q_4 so that the real cross ratio $|[q_1, q_2, q_3,, q_4)]|$ is less than 1. Then f and g can be normalized by an element $h \in PU(1,2;\mathbf{C})$ as follows.*

(1) hfh^{-1} has fixed points 0, ∞,

(2) hgh^{-1} has fixed points at Cygan distance r and $1/r$ from 0, where

$$r = |[q_1, q_2, q_3,, q_4)]|^{1/4}.$$

Proof. As in the proof of Lemma 2, there exists an element h_1 in $PU(1,2;\mathbf{C})$ such taht $h_1(q_1) = 0$ and $h_1(q_2) = \infty$. Take a complex dilation h_2 with its dilation factor $\{\delta(0, h_1(q_3)\delta(0, h_1(q_4)\}^{-1/2}$. Set $h = h_2 h_1$. Then it follows that $hfh^{-1}(0) = 0$, $hfh^{-1}(\infty) = \infty$. Also we see that hgh^{-1} has fixed points $h_2h_1(q_3)$, $h_2h_1(q_4)$. We have

$$\delta(0, h_2(h_1(q_3)))^4 = \frac{\delta(0, h_1(q_3))^2}{\delta(0, h_1(q_4))^2}$$

and

$$\delta(0, h_2(h_1(q_4)))^4 = \frac{\delta(0, h_1(q_4))^2}{\delta(0, h_1(q_3))^2}.$$

Using these, we obtain

$$\begin{aligned}\frac{\delta(0, h_1(q_3))^2}{\delta(0, h_1(q_4))^2} &= \frac{\delta(0, h_1(q_3))^2 \delta(h_1(q_4, \infty))^2}{\delta(0, h_1(q_4))^2 \delta(h_1(q_3, \infty))^2} \\ &= \frac{\delta^2(q_3, q_1)\delta^2(q_4, q_2)}{\delta^2(q_4, q_1)\delta^2(q_3, q_2)} \\ &= |[q_1, q_2, q_3,, q_4]| = r^4.\end{aligned}$$

Thus $\delta(0, h_2(h_1(q_3))) = r$ and $\delta(0, h_2(h_1(q_4))) = 1/r$.

Lemma 4. *Let f and g be loxodromic elements of $PU(1,2;\mathbf{C})$ with fixed points a_f, r_f, a_g, r_g, respectively. If there exists a stable basin point (r, r, ε) such that*

$$|[a_f, r_f, a_g, r_g]| < r^4,$$

and

$$\max\{|\lambda(f) - 1|, |\lambda(g) - 1|\} < \varepsilon,$$

then either f and g commute, or the group $< f, g >$ is not discrete.

Proof. Suppose that f and g do not commute. Therefore we may assume that f has an attracting fixed point 0 and a repelling fixed point ∞. By Lemma 3, it is possible to normalize so that a_g is in B_r and r_g is in $\overline{B}^c_{1/r}$. Consider the sequence of f-conjugates

$$g_1 = gfg^{-1}, g_2 = g_1 f g_1^{-1}, \ldots, g_k = g_{k-1} f g_{k-1}^{-1}, \ldots$$

Note that the fixed points of g_k are $a_k = g_{k-1}(0)$ and $r_k = g_{k-1}(\infty)$. We shall prove by induction that the sequence $\{g_k\}$ are distinct and contained in $S(r,\varepsilon)$. It is clear that f and g are contained in $S(r,\varepsilon)$. Since (r,r,ε) is a stable basin point, Lemma 2 implies that $a_{g_1} = g(0) \in B_r$ and $r_{g_1} = g(\infty) \in \overline{B}^c_{1/r}$. Noting that $\lambda(g_1) = \lambda(f)$, we see that g_1 is an element in $S(r,\varepsilon)$. Since $g(0) \neq 0$, f and g_1 are distinct. Now assume that $g_1, g_2, ..., g_k \in S(r,\varepsilon)$ are distinct with $\lambda(g_i) = \lambda(f)$ and with fixed points $\{a_{g_i}\}$ having the property that $\delta(a_{g_i}, 0)$ is minimal in the fixed point set of g_i. Moreover, assume that $\delta(a_{g_{i+1}}, 0) < \delta(a_{g_i}, 0)$ for $i = 1, 2, ..., k-1$. By Lemma 2, $\delta(g_k(0), 0) = \delta(a_{g_{k+1}}, 0) < \delta(a_{g_k}, 0)$ and $r_k \in \overline{B}^c_{1/r}$. Hence it follows by induction that all the $\{g_k\}$ are distinct and contained in $S(r,\varepsilon)$. Since B_r and $\overline{B}^c_{1/r}$ have disjoint, compact closures, there exists a subsequence $\{g_{k_l}\}$ such that $\{a_{k_l}\} \to a_\infty$ and $\{r_{k_l}\} \to r_\infty \neq a_\infty$. Noting that a loxodromic element is determined by its dilation factor and two fixed points, we conclude that $\{g_k\} \to g_\infty$ in $PU(1,2;\mathbf{C})$, where g_∞ is the unique element with fixed points a_∞ and r_∞ and $\lambda(g_\infty) = \lambda(f)$. Thus the group $< f, g >$ is not discrete.

Lemma 5. *Let g be a parabolic element with its fixed point ∞. Let $q \in H_3$ with $\delta(0,q) > \delta(0,g(0))$. Then*

$$|[0,q,g(0),g(q)]|^{\frac{1}{2}} \leq \left(1 + \frac{\delta(0,q)}{\delta(0,q) - \delta(0,g(0))}\right)\left(\frac{\delta(0,g(0))}{\delta(0,q) - \delta(0,g(0))}\right).$$

Proof. It follows from the triangle inequality that

$$\delta(g(0),q) \geq \delta(0,q) - \delta(0,g(0)).$$

Since $\delta(0,g(0)) = \delta(0,g^{-1}(0))$ and g is an isometry,

$$\delta(0,g(0)) = \delta(0,g^{-1}(0)) \geq \delta(0,q) - \delta(0,g(0)).$$

The triangle inequality also implies

$$\delta(q,g(q)) \geq \delta(0,q) - \delta(0,g(q)).$$

Hence we obtain

$$\begin{aligned}
|[0,q,g(0),g(q)]|^{\frac{1}{2}} &= \frac{\delta(0,g(0))\delta(q,g(q))}{\delta(0,g(q))\delta(q,g(0))} \\
&\leq \left(\frac{\delta(q,g(q))}{\delta(0,g(q))}\right)\left(\frac{\delta(0,g(0))}{\delta(0,q) - \delta(0,g(0)}\right) \\
&\leq \left(\frac{\delta(0,g(q)) + \delta(0,q)}{\delta(0,g(q))}\right)\left(\frac{\delta(0,g(0))}{\delta(0,q) - \delta(0,g(0))}\right) \\
&\leq \left(\frac{\delta(0,q)}{\delta(0,g(q))} + 1\right)\left(\frac{\delta(0,g(0))}{\delta(0,q) - \delta(0,g(0)}\right) \\
&\leq \left(\frac{\delta(0,q)}{\delta(0,g(q)) - \delta(0,g(0))} + 1\right)\left(\frac{\delta(0,g(0))}{\delta(0,q) - \delta(0,g(0)}\right).
\end{aligned}$$

We are ready to prove Theorem 1.

Proof of Theorem 1.

If $q=\infty$, then the group $< f,g >$ generated by f and g is not discrete. Therefore we assume that q is a finite point. Our assumption implies that $\delta(0,q)>\delta(0,g(0))$. Using Lemma 5, we have

$$\begin{aligned}|[0,q,g(0),g(q)]|^{\frac{1}{2}} &\leq \left(1+\frac{\delta(0,q)}{\delta(0,q)-\delta(0,g(0))}\right)\left(\frac{\delta(0,g(0))}{\delta(0,q)-\delta(0,g(0))}\right)\\ &\leq r^2.\end{aligned}$$

Set $h=gfg^{-1}$. We see that the fixed points of h are $g(0)$ and $g(q)$ and that the dilation factor of h is equal to that of f. It is clear that the fixed points of f and h are distinct. Hence $fh\neq hf$. By Lemma 4, the group $< f,h >$ is not discrete. Thus $< f,g >$ is not discrete.

3. Parker [5] gave a similar condition for a subgroup of $PU(1,n;\mathbf{C})$ to be discrete. The author would like to discuss the relation between Theorem 1 and Parker's Theorem in a subsequent paper.

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